European Journal of Operational Research 80 (1995) 166–174
North-Holland

Theory and Methodology

A discretizing algorithm for location problems

E.J. Carrizosa and J. Puerto

Departamento de Estadística e I.O., Universidad de Sevilla, Sevilla, Spain

Received August 1991; revised April 1993

Abstract: A new and simple methodology is proposed to solve both constrained and unconstrained planar continuous single-facility location problems. As particular instances, the classical location problems with mixed gauges can be solved. Theoretical convergence is proved, and numerical examples are given, showing a fast convergence in a small number of steps.

Keywords: Location; Facilities; Nonlinear programming

1. Introduction

In this paper, we study different problems within the scope of Planar Location Theory. The single-facility location problem may be generally defined as follows:

Given a set of individuals or collectivities (demand points), the localization of which is known, the location of a service facility must be chosen in a way that best satisfies the needs of the collectivity.

Depending on the exact meaning given to the word *best*, this general formulation will produce different problems. Our goal is to find an algorithm applicable to quite general statements, that is: allowing a certain degree of freedom in the selection of the globalizing function and the distance measures in the problem.

Correspondence to: Dr. J. Puerto, Departamento de Estadística e I.O., Universidad de Sevilla, Sevilla, Spain.

We propose the following formulation:

(P)

$$\begin{array}{ll} \min & F(x)=\phi(\gamma_1(x-a_1),\dots,\gamma_n(x-a_n)) \\ \text{s.t.} & x\in X, \end{array}$$

where $A=\{a_1,\dots,a_n\}$ is the set of demand points; $\phi:\mathbb{R}^n\to\mathbb{R}$ is the globalizing function; $\gamma_i(x-a_i)$, the function that measures the distance from a_i to x, is a gauge (Rockafellar, 1970), and X is the feasible set, which is supposed to be nonempty, convex and closed.

Remark that

(i) The functions that measure distances are assumed to be gauges, which is much more general than considering norms (the symmetric gauges); hence, we can model asymmetric situations where the distance from one point to another may differ from the distance back (rush-hour traffic, flight in the presence of wind, navigation in presence of water currents, inclined

SSDI 0377-2217(93)E0145-N

terrains, etc.; see, e.g., Drezner and Wesolowsky, 1989; Plastria, 1992).

(ii) Such gauges are not assumed to be the same for the different demand points. As Hansen et al. (1980) point out, there are three main reasons to use mixed gauges: it is often profitable for the facility to use different transportation modes for carrying its commodities (coal by water, and steel by rail); besides, it could be the case that some transportation modes are available only in some regions. Finally, variations in topography may lead to better fits by associating different gauges to different sites.

In the literature, some results for particular cases of (P) can be found. For example, Hansen et al. (1985) solve a generalized minisum problem by means of Branch and Bound techniques; Michelot and Lefebvre (1987) give a primal–dual algorithm for the unconstrained Weber problem with mixed gauges; Plastria (1987) proposes a cutting-plane algorithm for convex and nondecreasing globalizing functions; Juel and Love (1988) construct a set that, under certain conditions of symmetry, contains an optimal solution of the minisum problem with mixed norms.

In order to solve problem (P), we propose in this paper an algorithm that only needs the numerical evaluation of the objective function at the inspected points (thus avoiding subdifferential calculus), and, thanks to its special nature, it produces rather good approximated solutions quite quickly.

2. The lattice L_δ

The Discretizing Algorithm for Location Problems (DALP) we propose needs two starting feasible points, on which we will construct a certain type of grid. Given two different points x^1, $x^2 \in X$, let $u = x^2 - x^1$, and v a nonzero vector, orthogonal to u. For $\delta > 0$, we define the set

$$L_\delta = \{x \in \mathbb{R}^2 \colon x = x^1 + \delta \cdot , \cdot u + \delta \cdot n \cdot v \text{ for some } m.\ n \in \mathbb{Z}\}. \qquad (1)$$

An example is shown in Figure 1.

It should be noted that the affine geometrical properties of the lattice $\mathbb{Z}^2$ also hold in L_δ, as a consequence of the fact that L_δ is just $\mathbb{Z}^2$ after a change of base in $\mathbb{R}^2$. We shall look over those that will be used in the algorithm.

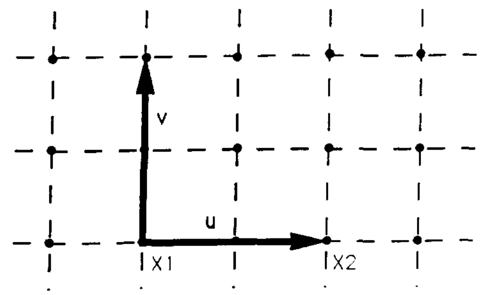

Figure 1. The lattice $L_{1/2}$.

Let A be the matrix whose columns are the vectors u and v.

Property 1. *The equation of a straight line r passing through 2 points of L_δ can be written as*

$$(a, b)' A^{-1}(x - x^1) = \delta c,$$

for some integers a, b and c, with $a \geq 0$, and a and b relatively prime.

Property 2. *The lines of equations*

$$(a, b)' A^{-1}(x - x^1) = \delta(c + 1)$$

and

$$(a, b)' A^{-1}(x - x^1) = \delta(c - 1)$$

are parallel to r, and no point between either one of them and r belongs to L_δ.

Property 3. *Let y^1 and y^2 be two adjacent points of r (that is: $L_\delta \cap [y^1, y^2] = \{y^1, y^2\}$), and $m \in \mathbb{Z}^2$ such that*

$$m_1 \cdot a + m_2 \cdot b = 1$$

(Bezout's identity); then, the points $y^1 + m$ and $y^2 + m$ (resp. $y^1 - m$ and $y^2 - m$) are adjacent points of

$$(a, b)' A^{-1}(x - x^1) = \delta(c + 1)$$

(resp. $(a, b)' A^{-1}(x - x^1) = \delta(c - 1)$).

The proofs are a straightforward extension of those given in Hoffman and Wolfe (1985), after an appropriate change of reference system.

A graphical illustration of these properties is given in Figure 2.

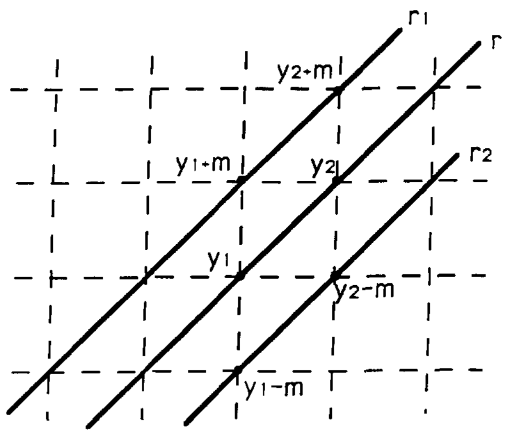

Figure 2. Properties of L_δ.

In this example with $x^1 = (0, 0)$, $x^2 = (1, 0)$, $v = (0, 1)$ and $\delta = 1$; it follows that

$$A = A^{-1} = I$$

(the identity matrix).

The line r through $y^1 = (1, 0)$, $y^2 = (2, 1)$ is

$$(1, -1)Ix = 1.$$

The lines r^1, r^2 with equations resp.

$$(1, -1)Ix = 2 \text{ and } (1, -1)Ix = 0$$

are parallel and adjacent to r.

Finally the vector $m = (1, 0)$ verifies Bezout's identity, thus the points $y^1 + m$, $y^2 + m$ (resp. $y^1 - m$, $y^2 - m$) verify Property 3.

3. An extension of Hoffman–Wolfe's algorithm for unimodal functions

Hoffman–Wolfe's algorithm (Hoffman and Wolfe, 1985), minimizes *unimodal* functions over $\mathbb{Z}^2$. We use this algorithm in order to allow the minimization of F on the grids L_δ, what will constitute a single step in our DALP.

Hoffman and Wolfe define *unimodality* as follows:

A function $f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said unimodal if for all x^0, x^1, x^2, x^0 in the segment with extremes x^1 and x^2, $f(x^1)$ finite, and $f(x^1) \leq f(x^0)$ imply $f(x^0) \leq f(x^2)$. Thus, once f is nondecreasing in a direction, it remains so.

When f takes finite values only, unimodality is equivalent to explicit quasiconvexity (i.e.: both quasiconvexity and strict quasiconvexity; see Martos, 1975), as shown below:

Theorem 1. *The function $f: \mathbb{R}^n \to \mathbb{R}$ is unimodal iff it is explicitly quasiconvex.*

Proof. Let f be unimodal. We first prove that it is also quasiconvex. For x^1, $x^2 \in \mathbb{R}^n$, and $x^0 \in (x^1, x^2)$, suppose w.l.o.g. that

$$\max\{f(x^1), f(x^2)\} = f(x^1). \tag{2}$$

If $f(x^0) > f(x^1)$, then, using unimodality, $f(x^0) \leq f(x^2)$, so $f(x^1) < f(x^2)$, which contradicts (2). We now prove that f is strictly quasiconvex: suppose that $f(x^1) > f(x^2)$. If $f(x^0) \geq f(x^1)$, the same argument shows that $f(x^1) \leq f(x^0) \leq f(x^2)$, which is a contradiction.

Conversely, let f be a explicitly quasiconvex function that is not unimodal. Then, there exist x^0, x^1, x^2 such that $x^0 \in (x^1, x^2)$, $f(x^1) \leq f(x^0)$, $f(x^2) < f(x^0)$. If $f(x^1) \neq f(x^2)$, then, as f is explicitly quasiconvex,

$$f(x^0) < \max\{f(x^1), f(x^2)\},$$

which is absurd. If $f(x^1) = f(x^2)$, then, using the quansiconvexity of f,

$$f(x^0) \leq \max\{f(x^1), f(x^2)\} = f(x^1) = f(x^2),$$

which, again, is absurd. Hence, f is unimodal. □

Consider the problem

$$\min \quad \{f(x): x \in X \cap L_\delta\},$$

where f is a explicitly quasiconvex function and X is a nonempty closed convex set in $\mathbb{R}^2$. The function is supposed to take a value M (M big enough) in $\mathbb{R}^2 \setminus X$.

We now outline Hoffman–Wolfe's algorithm on grids L_δ.

This algorithm needs two starting points x^1 and $x^2 \in L_\delta$ or, equivalently, a line r such that

$$\mathrm{card}(L_\delta \cap X \cap r) \geq 2.$$

The steps of this algorithm are the following:

Step 1. Find the two best points y^1 and y^2 of f on $L_\delta \cap X \cap r$.

Step 2. Let r^1 be one of the two adjacent lines to r.
Find the best point z of f on the set $L_\delta \cap X \cap r^1$.
If z is not better than y^1 or y^2, then go to Step 3;
else
set r the line through z and the best point among y^1 and y^2;
go to Step 1.

Step 3. Let r^2 be the other adjacent line to r.
Find the best point z of f on the set $L_\delta \cap X \cap r^2$.
If z is not better than y^1 or y^2, then stop: y^1 and y^2 are the two best points;
else
set r the line through z and the best point among y^1 and y^2;
go to Step 1.

4. Discretization of the problem (P)

In order to obtain operative results for the general Single-Facility Location Problem (P), some restrictions are imposed on the globalizing function ϕ. In the sequel we assume that ϕ verifies:

a) The function ϕ is a norm in $\mathbb{R}^n$.

b) ϕ is nondecreasing, that is: $\phi(\boldsymbol{u}) \le \phi(\boldsymbol{v})$ for all $\boldsymbol{u} = (u_1, u_2, \dots, u_n)$, $\boldsymbol{v} = (v_1, v_2, \dots, v_n)$ such that $0 \le u_i \le v_i \quad \forall i = 1, \dots, n$.

This condition is verified in many continuous single location problems with mixed gauges in the literature; for instance,

$$\phi(\boldsymbol{u}) = \sum_1^n w_i u_i$$

reduces to the Weber problem,

$$\phi(\boldsymbol{u}) = \max\{w_i u_i : 1 \le i \le n\}$$

to the Rawls problem, and

$$\phi(\boldsymbol{u}) = \alpha\left(\sum_1^n w_i u_i\right) + (1-\alpha)\max\{w_i u_i : 1 \le i \le n\}$$

to the cent-dian problem (Halpern, 1978). The latter also generates the efficient set for the bicriteria problem

$$\min\left(\sum w_i \gamma_i(x - a_i), \max\{w_i \gamma_i(x - a_i) : 1 \le i \le n\}\right)$$

(Hansen and Thisse, 1981), by varying the parameter α.

General results

Given two points x^1 and $x^2 \in \mathbb{R}^2$, we have defined the grids L_δ. For $\delta > 0$, let the constrained problem (P_δ) be

(P_δ)

$$\begin{array}{ll} \min & F(x) = \phi(\gamma_1(x - a_1), \dots, \gamma_n(x - a_n)) \\ \text{s.t.} & x \in X \cap L_\delta, \end{array}$$

and denote respectively by $v(\mathrm{P}_\delta)$ and S_δ its optimal value and the set of optimal solutions.

We now define a norm intimately related to L_δ. In terms of this norm, we first construct a rectangle that contains the set S of optimal solutions of (P), and, besides, we compare v(P) and $v(\mathrm{P}_\delta)$. As main consequence, we will be able to find, in terms of δ, an upper bound of the error committed it we take $v(\mathrm{P}_\delta)$ as the optimal value of (P), and hence, we can determine how small δ must be taken if we wish a certain accuracy in the estimation of v(P).

For nonzero orthogonal vectors $u, v \in \mathbb{R}^2$, we denote by $N^{u,v}$ the polyhedral norm whose unit ball has as vertices $u + v$, $u - v$, $-u + v$, $-u - v$, that is:

$$N^{u,v}(\alpha u + \beta v) = \max\{|\alpha|, |\beta|\}.$$

Let also $B^{u,v}$ be the balls of $N^{u,v}$:

$$B^{u,v}(x; \rho) = \{z \in \mathbb{R}^2 : N^{u,v}(x - z) \le \rho\},$$

and

$$F^t(x) = \phi(\gamma_1(a_1 - x), \dots, \gamma_n(a_n - x)).$$

Obviously, F^t coincides with F when the gauges γ_i are norms, because they are then symmetrical functions.

First of all, we mention how to construct a rotated rectangle that contains the set S of optimal solutions of (P).

Theorem 2. *There exist* $\varepsilon_1, \dots, \varepsilon_n \in (0, +\infty)$ *such that*

$$S \subset \bigcap_{x \in X} B^{u,v}\left(x; \frac{F(x) + F^t(x)}{\phi(1/\varepsilon_1, \dots, 1/\varepsilon_n)}\right).$$

Proof. For given gauges γ and γ^*, the function γ/γ^* is bounded from above in $\mathbb{R}^2 \setminus \{0\}$ and bounded from below by a positive number.

Then, for $i = 1, \ldots, n$, let

$$\varepsilon_i = \sup_{z \neq 0} \{N^{u,v}(z)/\gamma_i(z)\}.$$

Hence, $\varepsilon_i \in (0, +\infty)$ for all i. Let $x^* \in S$ and $x \in X$; it follows that

$$\begin{aligned}\frac{N^{u,v}(x^*-x)}{\varepsilon_i} &\leq \gamma_i(x^*-x)\\ &\leq \gamma_i(x^*-a_i)\\ &\quad + \gamma_i(a_i - x) \quad \text{for all } i.\end{aligned}$$

Hence,

$$\begin{aligned}N^{u,v}(x^*-x)\phi\left(\frac{1}{\varepsilon_1}, \ldots, \frac{1}{\varepsilon_n}\right) &\leq F(x^*) + F^t(x)\\ &\leq F(x) + F^t(x).\end{aligned}$$

Varying x on X, the theorem holds. □

In some particular cases, the scalars ε_i can easily be determined.

Example 1. If the unit ball of γ_i has u and v as symmetry axes, that is,

$$\gamma_i(\alpha u + \beta v) = \gamma_i(|\alpha| u + |\beta| v)$$

for all $\alpha, \beta \in \mathbb{R}$, then, it follows that

$$\varepsilon_i \leq \left(\min\{\gamma_i(u), \gamma_i(v)\}\right)^{-1}.$$

Example 2. If γ_i is a polyhedral gauge, that is, its unit ball is polyhedral with V as set of vertices, with the arguments of Lemma 1 below, it can be seen that

$$\varepsilon_i = \max\{N^{u,v}(d): d \in V\}.$$

For the general case, the constants ε_i could be calculated, if necessary, as

$$\begin{aligned}\varepsilon_i^{-1} = \min\Big\{ &\min_{t\in[0,1]} \gamma_i((1-t)P_1 + tP_2),\\ &\min_{t\in[0,1]} \gamma_i((1-t)P_2 + tP_3),\\ &\min_{t\in[0,1]} \gamma_i((1-t)P_3 + tP_4),\\ &\min_{t\in[0,1]} \gamma_i((1-t)P_4 + tP_1)\Big\}\end{aligned}$$

where $P_1 = u + v$, $P_2 = u - v$, $P_3 = -u + v$, $P_4 = -u + v$.

Remark that the four one-dimensional problems above are convex.

Example 3. Consider the situation depicted in Figure 3, with $X = \mathbb{R}^2$, $a_1 = (0, 0)$, $a_2 = (1, 1)$, $\gamma_1 = \|\cdot\|_1$ (the ℓ_1 norm), $\gamma_2 = \|\cdot\|_2$ (the Euclidean norm), and

$$\phi(u_1, u_2) = u_1 + u_2,$$

i.e. we are solving the unconstrained Weber problem

$$\min_{x\in\mathbb{R}^2} \|x - a_1\|_1 + \|x - a_2\|_2.$$

For $u = (1, 0)$ and $v = (0, 1)$, as

$$\varepsilon_1 = \sup_{z\neq 0} \frac{N^{u,v}(z)}{\gamma_1(z)} = 1$$

and

$$\varepsilon_2 = \sup_{z\neq 0} \frac{N^{u,v}(z)}{\gamma_2(z)} = 1,$$

it follows that

$$\begin{aligned}S &\subseteq \bigcap_{x\in\mathbb{R}^2} B^{u,v}\left(x, \tfrac{1}{2}(F(x) + F^t(x))\right),\\ &\subseteq B^{u,v}\left(a_1, \sqrt{2}\right) \cap B^{u,v}(a_2, 2).\end{aligned}$$

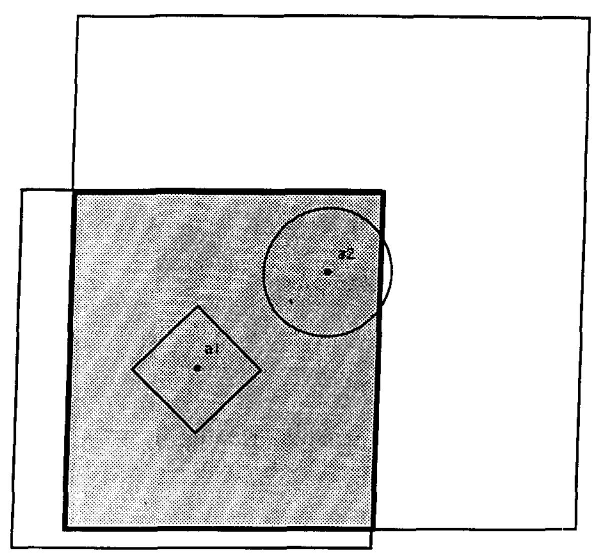

Figure 3. The superset given by Theorem 2.

In Figure 3, the dashed area represents the superset containing the solutionset S. Each demand point is represented with its unit ball.

In order to prove termination of the algorithm, it is supposed that there exists a constant K such that $\{x \in L_\delta : F(x) \le K\}$ is bounded and nonempty. This condition is satisfied once the restrictions a) and b) have been imposed on ϕ, because F has compact level sets.

Before comparing the problems (P) and (P_δ), one lemma is needed.

Let $G : \mathbb{R}^2 \to \mathbb{R}$,

$$G(z) = \phi(\gamma_1(z), \dots, \gamma_n(z))$$

for $z = (z_1, z_2) \in \mathbb{R}^2$, and

$$U(\phi; \gamma_1, \dots, \gamma_n) = \max\{G(u+v),\ G(u-v),\ G(-u+v),\ G(-u-v)\}.$$

Lemma 1. *For all* $z \in \mathbb{R}^2$,

$$G(z) \le U(\phi; \gamma_1, \dots, \gamma_n) N^{u,v}(z).$$

Proof. Let

$$\alpha = \max\left\{ \frac{G(z)}{N^{u,v}(z)} : z \neq 0 \right\} = \max\{G(z) : N^{u,v}(z) = 1\},$$

because ϕ and all γ_i are at least gauges. Besides, the boundary of the unit ball of $N^{u,v}$ is easily determined, so

$$\begin{aligned}\alpha = \max\{&\max\{G(u+tv) : |t| \le 1\},\\ &\max\{G(tu+v) : |t| \le 1\},\\ &\max\{G(-u+tv) : |t| \le 1\},\\ &\max\{G(tu-v) : |t| \le 1\}\}.\end{aligned}$$

These four functions are all convex, because gauges are convex, and ϕ is nondecreasing, so these four maxima are attained at the boundary of the domain, that is,

$$\alpha = \max\{G(u+v), G(u-v), G(-u+v), G(-u-v)\} = U(\phi; \gamma_1, \dots, \gamma_n). \quad \square$$

The following theorem gives upper and lower bounds for $v(\mathrm{P})$ in terms of $v(\mathrm{P}_\delta)$.

Theorem 3. *If* $X \cap L_\delta \neq \emptyset$, *then*

$$v(P_\delta) \ge v(P) \ge v(P_\delta) - U(\phi; \gamma_1, \dots, \gamma_n)\rho_\delta,$$

where

$$\rho_\delta = \sup_{x \in X} \inf_{y \in L_\delta \cap X} N^{u,v}(x-y).$$

Proof. The first inequality is trivial. For the second one, let $x^\delta \in S^\delta$ and $x^* \in S$; $\alpha = v(\mathrm{P}_\delta) - v(\mathrm{P})$. First, we can find a point $y^\delta \in L_\delta \cap X$ such that

$$N^{u,v}(y^\delta - x^*) \le \rho_\delta.$$

Hence, $\alpha \le F(y^\delta) - F(x^*)$. As the triangular inequality holds for gauges (Durier and Michelot, 1985), it follows that

$$\alpha \le \phi(\gamma_1(y^\delta - x^*) + \gamma_1(x^* - a^1), \dots, \gamma_n(y^\delta - x^*) + \gamma_n(x^* - a^n)) - F(x^*).$$

Furthermore, because of the fact that ϕ is subadditive, and using Lemma 1,

$$\begin{aligned}\alpha &\le F(x^*) + G(y^\delta - x^*) - F(x^*) = G(y^\delta - x^*)\\ &\le U(\phi; \gamma_1, \dots, \gamma_n) N^{u,v}(y^\delta - x^*)\\ &\le U(\phi; \gamma_1, \dots, \gamma_n)\rho_\delta. \quad \square\end{aligned}$$

In spite of its involved expression, the constant ρ_δ can be evaluated (at least overestimated) in some particle cases. For example, if $X = \mathbb{R}^2$, then it follows that $\rho_\delta \le \frac{1}{2}\delta$.

With the notation developed above, and a given stopping rule, the DALP may be described as follows:

Step 0. Take two starting points, x^1, $x^2 \in X$.
Take $\lambda > 0$ (the step).
Set $\delta = 1$.

Step 1. Set $u = x^2 - x^1$, and v a nonzero vector, orthogonal to u.
For any $\delta > 0$, define L_δ as in (1).
Using Hoffman–Wolfe's algorithm, find y^1 and y^2, the two best solutions on $X \cap L_\delta$.

If the stopping rule is satisfied, go to step 2;
else

Set $x^1 = y^1$;
Set $x^2 = y^2$;
Set $\delta = \delta/\lambda$;

Go to Step 1.

Step 2. Take the best of y^1 and y^2 as the optimal solution of (P), and its value as v(P). STOP.

Among the possible stopping rules (see, e.g., Bazaraa and Shetty, 1979), two rules seem to be particularly appropriate for this algorithm; as Theorem 3 gives an upper bound of the error committed, one might stop when

$$U(\phi; \gamma_1, \gamma_2, \ldots, \gamma_n)\rho_\delta < \varepsilon$$

for some prespecified $\varepsilon > 0$. On the other hand, one might stop when the width of the grid L_δ is small enough, i.e.: when $\delta < \delta_0$ for some prespecified $\delta_0 > 0$.

Remark that in the algorithm above the numerical evaluation of the objective function at the inspected points suffices, thus subdifferential calculus is not needed, which is a remarkable feature when subdifferential calculus is cumbersome.

Although it does not seem easy to find a scenario reflecting this situation without intricate mathematics, we can consider as an illustration the following dynamic model: it is a fact that travel times are time-varying (depending on traffic conditions, weather, etc.). Hence, if at each instance t the distance from demand point a_i to x is a gauge $\gamma_i(x - a_i, t)$, then the average distance in the interval time T is

$$\gamma_i^*(x - a_i) = \int_T \gamma_i(x - a_i, t)\, \mathrm{d}\mu_i(t),$$

where μ_i is a certain probability measure.

In general, the integrals above can only be numerically solved; while this can be done by standard software packages, the calculus of the subdifferential is a rather complex task.

Observe that, in this situation, the classical approach of fitting by a (weighted) ℓ_p norm (see, e.g., Berens, 1988) seems to be inappropriate.

The iterative nature of the algorithm reduces, by using at each step as starting points the two best points obtained in the previous step, the computational effort, compared with a one-step algorithm with a small δ.

The convergence of the procedure is studied in the next section.

5. Convergence of the algorithm

The theorem below assures the convergence of the DALP. Due to the fact that, in general, there is not only one optimal solution of (P), the strongest result we can state is the following:

Theorem 4. *Let $\{\delta_n\}$ be a sequence of positive real numbers converging to 0. For each n, let x^n be an optimal solution of the discretized problem* (P_{δ_n}). *Then, any accumulation point of the sequence $\{x^n\}$ is an optimal solution of* (P).

Proof. Let x be an accumulation point of the sequence $\{x^n\}$, and let $x^* \in S$, the set of optimal solutions for (P). For a given norm $\|\cdot\|$ and $\delta > 0$, denote by $B(x^*, \delta)$ the set $\{y \in \mathbb{R}^2 : \|x^* - y\| < \delta\}$.

As F is continuous at x^*, for a given $\varepsilon > 0$, there exists a $\delta_1 > 0$ such that

$$|F(x^*) - F(y)| < \varepsilon$$

for all y in $B(x^*, \delta_1)$. It can be easily seen that there exists a certain $n_0 \in \mathbb{N}$ such that

$$L_{\delta_n} \cap B(x^*, \delta) \cap X \neq \phi$$

for all $n \geq n_0$. For $n \geq n_0$, we can thus take y^n in $L_{\delta_n} \cap B(x^*, \delta_1) \cap X$; it follows that

$$F(x^n) \leq F(y^n) < \varepsilon + F(x^*)$$

for all $n \geq n_0$. Taking limits in n, and using the fact that F is continuous, we deduce that $x \in S$. □

When the set S of optimal solutions for (P) contains only one point, the theorem above can be improved.

Corollary 1. *If $S = \{x^*\}$, then the sequence $\{x^n\}$ defined in the preceding theorem converges to x^*.*

In order to assure the uniqueness of the solution of (P), some further conditions must be im-

Table 1
Problem 1: Minisum with ℓ_2 norms

Step	δ	Optimal point	Functional value	Inspections
1	1	(6.0000, 4.0000)	290.8704	72
2	10^{-1}	(5.6000, 4.1000)	288.6931	45
3	10^{-2}	(5.5700, 4.1500)	288.6658	126
4	10^{-3}	(5.5570, 4.1450)	288.6656	51
5	10^{-4}	(5.5689, 4.1471)	288.6656	26
6	10^{-5}	(5.5689, 4.1470)	288.6656	26

posed on ϕ and the gauges, as the ones studied by Pelegrín, Michelot and Plastria (1985).

6. Some examples

A set $A = \{a_1, \ldots, a_{20}\}$ of 20 demand points are taken. The coordinates of the points and the weights for the gauges are given in Plastria (1987, Table 1).

In our first example, we solve the unconstrained minisum problem with the ℓ_2-norm.

In the second example, the objective function is the same, but we restrict the feasible set to the rectangle $[0,5] \times [0,4]$. In particular, the optimal solution for problem 1 is not feasible for problem 2.

In the third example, the demand points and weights are the same as in previous examples; however, the gauge γ_i are different in different points: γ_i is the ℓ_1-norm for $i = 1, \ldots, 7$, γ_i is the ℓ_2-norm for $i = 8, \ldots, 15$, and otherwise γ_i is the polyhedral asymmetric gauge with extremal vertices (0, 1), (1, −1) and (−1, −1). The globalizing function ϕ for this problem is

$$\phi(u_1, \ldots, u_n) = 0.5 \cdot \sum_1^n u_i + 0.5 \cdot \max\{u_1, \ldots, u_n\},$$

Table 2
Problem 2: Minisum with ℓ_2 norms subject to $[0,5] \times [0,4]$

Step	δ	Optimal point	Functional value	Inspections
1	10^{-1}	(4.5000, 3.6000)	291.8268	38
2	10^{-2}	(4.9500, 3.9600)	291.8269	12
3	10^{-3}	(4.9950, 3.9960)	291.8269	12
4	10^{-4}	(4.9995, 3.9996)	291.8269	12
5	10^{-5}	(4.9999, 3.9999)	291.8269	12
6	10^{-6}	(5.0000, 4.0000)	291.8269	15

Table 3
Problem 3: Cent-dian ($\alpha = \frac{1}{2}$) with mixed gauges ℓ_1, ℓ_2, Δ

Step	δ	Optimal point	Functional value	Inspections
1	1	(4.00000, 3.00000)	179.6483	95
2	10^{-1}	(4.90000, 3.50000)	179.3786	30
3	10^{-2}	(4.99000, 3.59000)	179.3786	26
4	10^{-3}	(4.99900, 3.59900)	179.3786	26
5	10^{-4}	(4.99990, 3.59990)	179.3786	27
6	10^{-5}	(4.99999, 3.60000)	179.3786	27

that is, we are solving the unconstrained mixed-gauges cent-dian problem with $\alpha = 0.5$.

The first iterations for these examples are shown respectively in Tables 1, 2 and 3. In these tables, the columns represent respectively the step in the algorithm, the width (δ) of the grid, the optimal solution x^δ on L_δ, its functional value, and the number of inspected points in each step.

In all the examples, we consider $\lambda = 10$, and $x^1 = (0, 0)$, $x^2 = (1, 0)$ as starting points.

For the first one, we obtain in the fourth iteration the optimal solution given by Plastria (1987).

7. Conclusions

In this paper we have proposed a new approach for solving constrained and unconstrained planar single-facility location problems under quite general assumptions about the globalizing function. Our method avoids the calculus of subgradients, and, what is more, explicit forms for gauges are not needed.

Convergence to the optimal value is shown. Besides, the algorithm is very easy to implement; some examples are given, showing a fast convergence in a small number of iterations.

Acknowledgement

Thanks are due to Dr. Frank Plastria and two anonymous referees for their very constructive comments.

References

Bazaraa, M.S., and Shetty, C.M. (1979), *Nonlinear programming. Theory and Algorithms*, Wiley, New York.

Berens, W. (1988). "The suitability of the weighted ℓ_p-norm in estimating actual road distances", *European Journal of Operational Research* 34, 39–43.

Drezner, Z., and Wesolowsky, G.O. (1989), "The asymmetric distance location problems", *Transportation Science* 23, 201–207.

Durier, R., and Michelot, C. (1985), "Geometrical properties of the Fermat–Weber problem", *European Journal of Operational Research* 20, 332–343.

Halpern, J. (1978), "Finding minimal center-median convex combination (cent-dian) of a graph", *Management Science* 34, 535–544.

Hansen, P., and Thisse, J.F. (1981), "The generalized Weber–Rawls problem", in J.P. Brans (ed.), *Operations Research*, North-Holland, Amsterdam.

Hansen, P., Perreur, J., Thisse, J.F. (1980), "Location theory, dominance and convexity: Some further results", *Operations Research* 28, 1241–1250.

Hansen, P., Peeters, D., Richard, D., and Thisse, J.F. (1985), "The minisum and minimax location problems revisited", *Operations Research* 33, 1251–1265.

Hoffman, A.J., and Wolfe, P. (1985), "Minimizing a unimodal function of two integer variables", *Mathematical Programming Study* 25, 76–87.

Juel, H., and Love, R. (1988), "A localization property for facility-location problems with arbitrary norms", *Naval Research Logistics* 35, 203–207.

Martos, B. (1975), *Nonlinear Programming: Theory and Methods*, North-Holland, Amsterdam.

Michelot, C. and Lefebvre, O. (1987), "A primal-dual algorithm for the Fermat–Weber problem involving mixed gauges", *Mathematical Programming* 39, 319–335.

Pelegrín, B. Michelot, C., and Plastria, F. (1985), "On the uniqueness of optimal solutions in continuous location theory", *European Journal of Operational Research* 20, 327–331.

Plastria, F. (1987), "Solving general continuous single-facility location problems by cutting planes", *European Journal of Operational Research* 29, 98–110.

Plastria, F. (1992), "On destination optimality in asymmetric distance Fermat–Weber problems", *Annals of Operations Research* 40, 355–369.

Rockafellar, T. (1970), *Convex Analysis*, University Press, Princeton, NJ.